Show all of your work as partial credit will be given.

1. Explicitly solve the initial value problem

$$\frac{dy}{dx} + 4y - e^{-x} = 0, \ y(0) = \frac{4}{3}.$$

2. Demonstrate that the following equation is exact and then solve it

$$\left(\frac{1}{y} + e^x\right) dx - \frac{x}{y^2} dy = 0.$$

Solutions

Problem 1.

(i) This is a linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ with

$$P(x) = 4, \ Q(x) = e^{-x}.$$

(ii) The integrating factor is

$$\mu(x) = e^{\int P(x)dx} = e^{4x}.$$

(iii) The general solution is

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] = \frac{1}{e^{4x}}\left[\int e^{4x}e^{-x}dx + C\right] = \frac{1}{e^{4x}}\left[\int e^{3x}dx + C\right] = \frac{1}{e^{4x}}\left[\frac{1}{3}e^{3x} + C\right]$$

(iv) Using initial conditions,

$$\frac{4}{3} = \frac{1}{e^0}\left[\frac{1}{3}e^0 + C\right] = \frac{1}{3} + C.$$

Therefore, $C = 1$, and the solution of the I.V.P. is

$$y(x) = \frac{1}{3}e^{-x} + e^{-4x}.$$

Problem 2.

(i) First, we write the equation as $Mdx + Ndy = 0$, where

$$M(x, y) = \frac{1}{y} + e^x, \ N(x, y) = -\frac{x}{y^2}.$$

(ii) We find that

$$\frac{\partial M(x, y)}{\partial y} = -1/y^2, \ \frac{\partial N(x, y)}{\partial x} = -1/y^2$$

The derivatives are equal, therefore the equation is exact.

To solve the equation, we can start either with M or with N. In the first case,

$$F = \int M dx + g(y) = \int (\frac{1}{y} + e^x) dx + g(y) = \frac{x}{y} + e^x + g(y).$$

Therefore, the equality $\frac{\partial F(x,y)}{\partial y} = N(x,y)$ can be rewritten as

$$-\frac{x}{y^2} + g'(y) = -\frac{x}{y^2}$$

from which we find that function g is a constant, $g = c$. Then solutions satisfy to

$$\frac{x}{y} + e^x = c$$

If we start with N, then

$$F = \int N dy + h(x) = \int (-\frac{x}{y^2}) dy + h(x) = \frac{x}{y} + h(x).$$

Therefore, the equality $\frac{\partial F(x,y)}{\partial x} = M(x,y)$ can be rewritten as

$$\frac{1}{y} + h'(x) = \frac{1}{y} + e^x$$

from which we find that $h(x) = e^x$. Then solutions satisfy to

$$\frac{x}{y} + e^x = c.$$