Show all of your work as partial credit will be given.

1. Explicitly solve the initial value problem
\[
\frac{dy}{dx} + 4y - e^{-x} = 0, \quad y(0) = \frac{4}{3}.
\]

2. Demonstrate that the following equation is exact and then solve it
\[
\left(\frac{1}{y} + e^x\right) dx - \frac{x}{y^2} dy = 0.
\]

Solutions

Problem 1.
(i) This is a linear differential equation \(\frac{dy}{dx} + P(x)y = Q(x)\) with
\[P(x) = 4, \quad Q(x) = e^{-x}.
\]
(ii) The integrating factor is
\[\mu(x) = e^{\int P(x)dx} = e^{4x}.
\]
(iii) The general solution is
\[y(x) = \frac{1}{\mu(x)} \left[\int \mu(x)Q(x)dx + C\right]
= \frac{1}{e^{4x}} \left[\int e^{4x}e^{-x}dx + C\right]
= \frac{1}{e^{4x}} \left[\int e^{3x}dx + C\right]
= \frac{1}{3} e^{3x} + C.
\]
(iv) Using initial conditions,
\[\frac{4}{3} = \frac{1}{e^{0}} \left[\frac{1}{3} e^{0} + C\right] = \frac{1}{3} + C.
\]
Therefore, \(C = 1\), and the solution of the I.V.P. is
\[y(x) = \frac{1}{3} e^{-x} + e^{-4x}.
\]

Problem 2.
(i) First, we write the equation as \(Mdx + Ndy = 0\), where
\[M(x, y) = \frac{1}{y} + e^x, \quad N(x, y) = -\frac{x}{y^2}.
\]
(ii) We find that
\[\frac{\partial M(x, y)}{\partial y} = -1/y^2, \quad \frac{\partial N(x, y)}{\partial x} = -1/y^2.
\]
The derivatives are equal, therefore the equation is exact.
To solve the equation, we can start either with $M$ or with $N$. In the first case,

$$F = \int M\,dx + g(y) = \int \left( \frac{1}{y} + e^x \right)dx + g(y) = \frac{x}{y} + e^x + g(y).$$

Therefore, the equality $\frac{\partial F(x,y)}{\partial y} = N(x, y)$ can be rewritten as

$$-\frac{x}{y^2} + g'(y) = -\frac{x}{y^2}$$

from which we find that function $g$ is a constant, $g = c$. Then solutions satisfy to

$$\frac{x}{y} + e^x = c$$

If we start with $N$, then

$$F = \int N\,dy + h(x) = \int \left( -\frac{x}{y^2} \right)dy + h(x) = \frac{x}{y} + h(x).$$

Therefore, the equality $\frac{\partial F(x,y)}{\partial x} = M(x, y)$ can be rewritten as

$$\frac{1}{y} + h'(x) = \frac{1}{y} + e^x$$

from which we find that $h(x) = e^x$. Then solutions satisfy to

$$\frac{x}{y} + e^x = c.$$