Transition to Optimal Control of ODEs, PDEs, Discrete and ID Models

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Why, space?

Heterogeneous environment
$*$ Or even if environment is homogeneous, population may have heterogeneous initial distribution.
Organisms may spread from regions of high density to less occupied habitat. Different species may have different rates of spread.

-dispersion-diffusion

Overview

To formulate an appropriate OC problem, start with a model which is a reasonable representation of the scenario to be considered.

– Decide how to represent the spatial and temporal aspects.

GOAL

Designing an appropriate objective functional is equally important.
– Balancing competing goals may be crucial.
(like eliminating a pest and minimizing cost of elimination)
– The form of the OC results depend strongly on the model and its parameters, how the controls enter that system, and the objective functional.

Optimal Control

Adjust controls in a system to achieve a goal
System:

- Ordinary differential equations
- Partial differential equations
- Discrete equations
- Inequalities and Matrix Equations
- Integro-difference equations

Discrete Models

-matrix models
-linear programming
-dynamic programming
-IBM individual based models
- CA cellular automata
— — — Note Andrew Whittle will discuss some examples of discrete models.

Matrix Models

$x_{n+1} = Mx_n$

where x is a vector and M is a matrix

$-----$ reference: H. Caswell, Matrix Population Models, Sinauer 2001

Cellular Automata

Kristen Bain's (and Nic Buchanan) fire model later will illustrate CA approach

IBM

Use patterns to design the processes in individual based model

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Individuals move in spatial grid by rules and transition probabilities

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Environmental features may change also

Linear Programming Approach

John Hof, Michael Bevers
Spatial Optimization for Managed Systems
–Cambridge Press 1998
– using cellular grids or geometric spaces to depict spatial options

–methods of linear programming, integer and nonlinear programming

Bevers and Hof, 2nd book

Spatial Optimization in Ecological Systems –Cambridge Press 2002

— — — explore formulations that capture highly nonlinear ecological effects with spatial linear programs that can be solved with simplex algorithms (and 2 integer-friendly programs)

model idea

$x_{i,t}$ state at location i, time t

dynamics

$x_{i,t+1} = f(x_{j,t}\text{neighbors}, u_{i,t}, s_{i,t})$

control u, other features s

$Ax_t \leq b$

$Bu_t \leq d$

$max \Sigma_{i,t} G(x_{i,t}, u_{i,t})$

Dynamic Programming

C W Clark and M.Mangel
Dynamic State Variable Models in Ecology,
Oxford 2000
— — — individual optimization models of behavior

dynamic state variable models = dynamic programming models

Formalizing constraints and trade-offs
KEY feature —–-Backward iteration
transition of states given decisions

Idea of Differential Equations

$\frac{x(t+\Delta t)-x(t)}{\Delta t}$

Difference quotient - rate of change of x with respect t

As Δt get small, the difference quotient approximates the instantaneous rate of change

$\rightarrow$

$x'(t)$ the derivative of x

Differential Equations

$x' = kx$
rate of change is proportional to the population x
exponential growth $x = x_0 e^{kt}$
Other types of growth:
$x' = rx(1-x)$ logistic growth
$x' = \frac{u}{1+u}x - Dx^2$ limited growth with input u

Deterministic Optimal Control

Control of Ordinary Differential Equations (DE)

$u(t)$ control

$x(t)$ state

State function satisfies DE

Control affects DE

$$x'(t) = g(t, x(t), u(t))$$

$u(t) \to x(t)$ Goal (objective functional)

Deterministic Optimal Control- ODEs

Find piecewise continuous control $u(t)$ and associated state variable $x(t)$ to maximize

$$\max \int_0^T f(t, x(t), u(t))\, dt$$

subject to

$$x'(t) = g(t, x(t), u(t))$$

$x(0) = x_0$ and $x(T)$ $free$