Optimal Harvesting of a Semilinear Elliptic Logistic Fishery Model

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Outline

- Development of Fishery Models
- Motivation
- The Model
- Optimal Control Problems
- Numerical Examples for $J_1$
- Numerical Examples for $J_2$
Development of Fishery Models[1]

- 1900 - 1920: First Efforts
  - F. I. Baranov: grandfather of fisheries population dynamics
  - ICES (1902): International Council for the Exploration of the Sea

1920 - 1960: Establishment of Science
- Ricker, Beverton and Holt, Leslie, Lotka and Volterra, Thompson etc.
- multi-species modeling,
- age- and size-structure dynamics;
1960 - 1980: Deterministic Theory, Statistical Practice

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- advances in age-structured models (*Gulland, Pope, Doubleday*),
- improvements to surplus production (*Pella, Tomlinson, Schnute, Fletcher, Hilborn*) and stock recruitment models,
- bioeconomic models (*Clark*)
- management control models (*Hilborn, Walters*)
• 1980-2000: The Golden Age
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  • integration between mathematics and statistics
  • Bayesian and time series methods (uncertainty)
  • realistic modeling for:
    • age and size-structured population
    • spatial dynamics
    • harvesting strategies (stochasticity, time variation)
• The New Millenium
  • future models:
    • habitat and spatial concerns
    • genetics
    • multispecies interactions
    • environmental factors
    • effects of harvesting on the ecosystem
    • socioeconomic concerns
Motivation

Neubert *(Ecology Letter, 2003)* studied the fishery management problem:

Maximize the yield

\[ J(h) = \int_0^l h(x)u(x) \, dx, \quad 0 \leq h(x) \leq h_{\text{max}} \]

Subject to

\[- \frac{d^2u}{dx^2} = u(1 - u) - h(x)u, \quad 0 < x < l, \]

\[ u(0) = u(l) = 0. \]
Neubert’s Results

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Neubert’s Results

- No-take marine reserves are always part of an optimal harvest designed to maximize yield;
- The sizes and locations of the optimal reserves depend on a dimensionless length parameter;
- For small values of this parameter, the maximum yield is obtained by placing a large reserve in the center of the habitat;
- For large values of this parameter, the optimal harvesting strategy is a spatial “chattering control” with infinite sequences of reserves alternating with areas of intense fishing;
Our Fishery Model

\begin{align*}
-\Delta u &= ru(1 - u) - h(x)u, & x \in \Omega, \\
       u &= 0, & x \in \partial \Omega.
\end{align*}

where \( u(x) \) is the fish density, \( r \) is the growth rate, \( h(x) \) is the harvesting depending on the location of fish, \( \Omega \in R^n \), smooth and bounded domain.
Optimal Control Problems

Goals:

- Maximizing the yield and minimizing the cost of fishing.

\[ J_1(h) = \int_{\Omega} h(x)u(x) \, dx - \int_{\Omega} (B_1 + B_2 h)h \, dx, \quad h \in U_1. \]

- Maximizing the yield and minimizing the variation of the fishing effort.

\[ J_2(h) = \int_{\Omega} h(x)u(x) \, dx - A \int_{\Omega} |\nabla h|^2 \, dx, \quad h \in U_2, \]
Optimality System I

state equation

\[
\begin{cases}
  -\Delta u = ru(1 - u) - h(x)u, & x \in \Omega, \\
  u = 0, & x \in \partial\Omega;
\end{cases}
\]

adjoint equation

\[
\begin{cases}
  -\Delta p - r(1 - 2u)p + hp = h, & x \in \Omega, \\
  p = 0, & x \in \partial\Omega;
\end{cases}
\]

characterization of optimal control

\[
h(x) = \min\left\{ \max\left\{ 0, \frac{u - pu - B_1}{2B_2} \right\}, 1 - \delta \right\}.
\]
Numerical Examples for $J_1$: 1-D case, $B_2$ effect

Set $B_1 = 0.1$, vary $B_2 = 0.5, 1.25, 2.5, 5, 10$
Numerical Examples for $J_1$: 1-D case, small $B_2$

Set $B_1 = 0$, vary $B_2 = 0.1, 0.05, 0.01$
Numerical Examples for $J_1$: 2-D case

B_1 = 0, B_2 = 1, r = 5, L = 2.5

B_1 = 0, B_2 = 1, r = 5, L = 2.5

fish density for $J_1(h)$

optimal harvesting for $J_1(h)$
Numerical Examples for $J_1$: 2-D case, $B_1$ effect
Numerical Examples for $J_1$: 2-D case, domain size effect

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Numerical Examples for $J_1$: 2-D case, small $B_2$

$B_1 = 0, B_2 = 0.05, r = 5, L = 2.5$

Fish density for $J_1(h)$

Optimal harvesting for $J_1(h)$
Optimality System II

state equation

\[
\begin{aligned}
-\Delta u &= ru(1 - u) - h(x)u, & x \in \Omega, \\
\frac{\partial u}{\partial x} &= 0, & x \in \partial \Omega;
\end{aligned}
\]

adjoint equation

\[
\begin{aligned}
-\Delta p - r(1 - 2u)p + hp &= h, & x \in \Omega, \\
p &= 0, & x \in \partial \Omega;
\end{aligned}
\]

characterization of optimal control

\[
\min \left\{ \max \left( pu - u - 2A\Delta h, h - (1 - \delta) \right), h - 0 \right\} = 0.
\]
Numerical Examples for $J_2$:

Vary $A = 1, 2.5, 5, 10$
Conclusion: in the long run

- If we want to maximize yield and minimize cost \( J_1 \), then increasing labor cost \( B_2 \) or fixed cost \( B_1 \) will decrease optimal harvesting.

- If we only want to maximize yield, then reserve is part of the optimal harvesting strategy.

- For \( J_1 \), the optimal benefit increases when domain size increases.

- If we want to maximize yield and minimize variation in fishing effort, then increasing \( A \) will reduce optimal harvesting.
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