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Optimal Vaccination Strategies for the Control of Rabies among Raccoons using a metapopulation model

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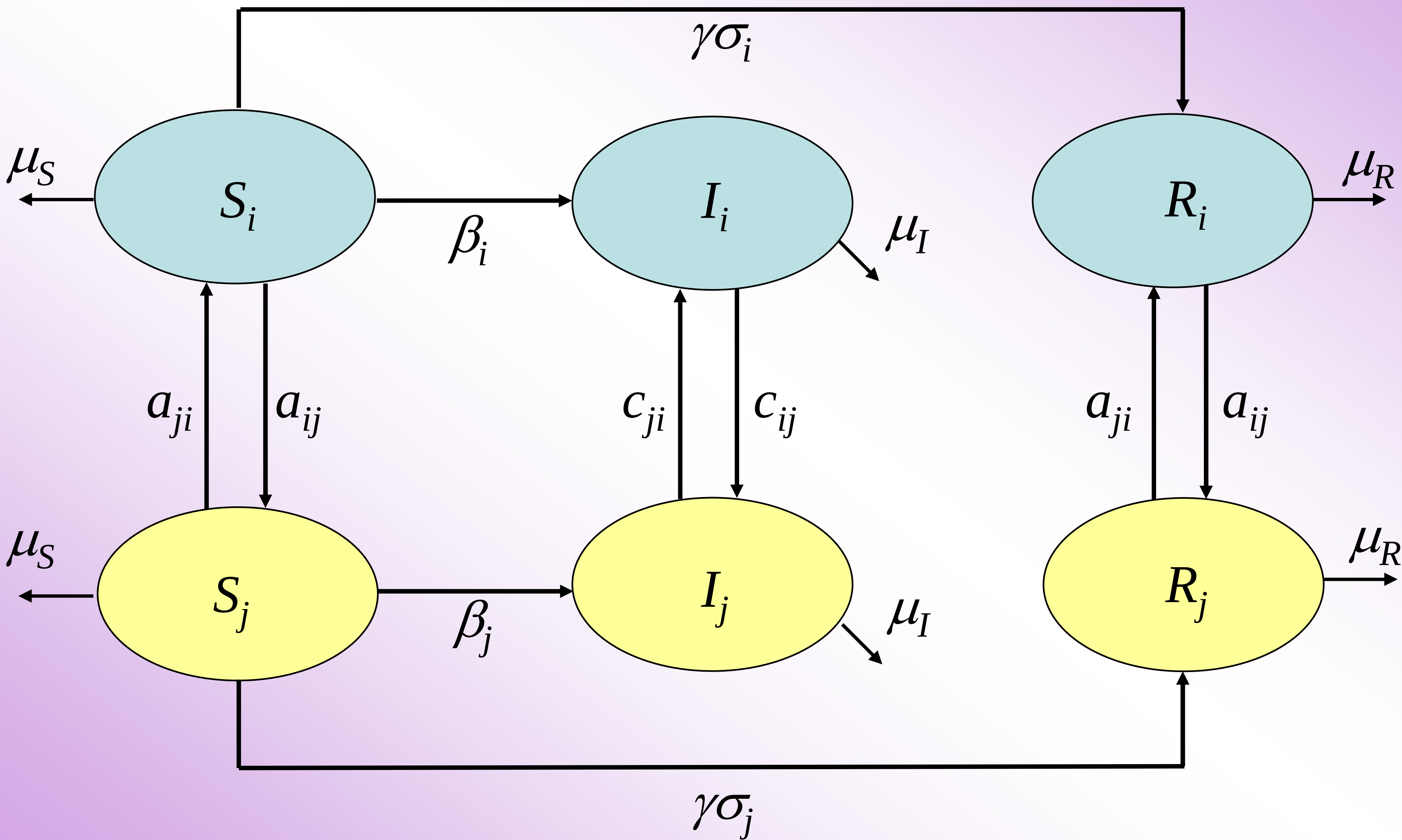
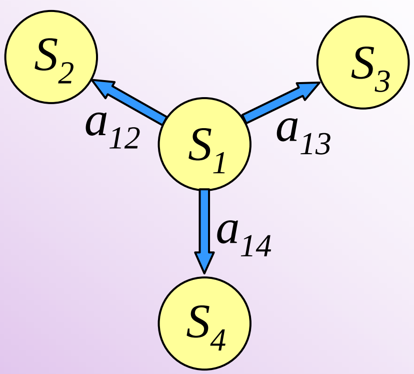
Flow Diagram

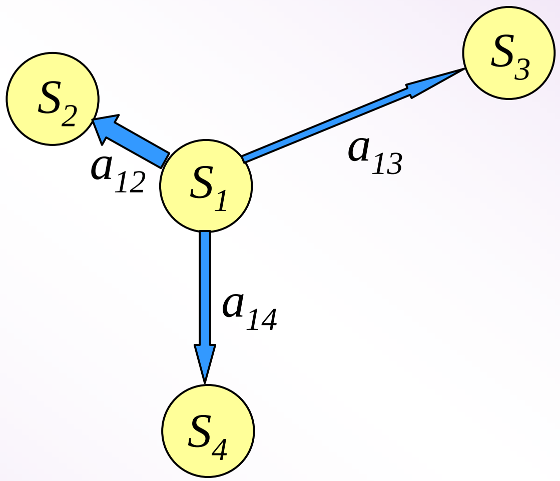


NOTE: The rate of transition a_{ij}, c_{ij} are chosen to represent spatial configuration and density-independent.

Examples



$a_{12} = a_{13} = a_{14}$



$a_{12} > a_{14} > a_{13}$

Notations

$$S(t) = (S_1(t), S_2(t), \cdots, S_n(t))$$

where $S_i(t)$ is the number of susceptibles in subpopulation i.

$$I(t) = (I_1(t), I_2(t), \cdots, I_n(t))$$

where $I_i(t)$ is the number of infecteds in subpopulation i.

$$R(t) = (R_1(t), R_2(t), \cdots, R_n(t))$$

where $R_i(t)$ is the number of individuals immune to the disease

in subpopulation i.

$$\mu(t) = (\mu_S, \mu_I, \mu_R)$$

$\mu(t)$ is the mortality rate in each class: S, I and R.

$$\beta(t) = (\beta_1, \beta_2, \cdots, \beta_n)$$

$\beta_i(t)$ is the rate of transmission in subpopulation i.

$$\sigma(t) = (\sigma_1(t), \sigma_2(t), \cdots, \sigma_n(t))$$

$\sigma_i(t)$ is the rate of removal(control) of susceptibles from subpopulation i to the immunes due to vaccinations.

γ is the efficacy of vaccination.

$A(t) = (n \times n)$ matrix

where the matrix element a_{ij} is the rate of transition of non-infecteds(susceptible and immune classes) from subpopulation i to subpopulation j.

$C(t) = (n \times n)$ matrix where the matrix element c_{ij} is the rate of transition of infecteds from subpopulation i to subpopulation j.

Base Model – No growth (birth)

$$(1)\quad \frac{dS_i}{dt} = -\beta_i S_i I_i - \gamma\sigma_i S_i + \sum_{j,j\neq i}^{n} a_{ji} S_j - \sum_{j,j\neq i}^{n} a_{ij} S_i - \mu_S S_i$$

$$(2)\quad \frac{dI_i}{dt} = \beta_i S_i I_i + \sum_{j,j\neq i}^{n} c_{ji} I_j - \sum_{j,j\neq i}^{n} c_{ij} I_i - \mu_I I_i$$

$$(3)\quad \frac{dR_i}{dt} = \gamma\sigma_i S_i + \sum_{j,j\neq i}^{n} a_{ji} R_j - \sum_{j,j\neq i}^{n} a_{ij} R_i - \mu_R R_i$$

ICs :

$$(4)\quad S_i(0) = S_{i0},\quad I_i(0) = I_{i0},\quad R_i(0) = R_{i0}$$

Control(Vaccination) :

$$(5)\quad 0 \le \sigma_i \le \sigma_{\max} \qquad \text{for } i = 1, 2, \ldots, n.$$

Objective Functional

We wish to minimize total number of infecteds and the cost associated with vaccination.

$$(6) \qquad Minimize \quad J(\sigma) = \sum_{i=1}^{n} \int_0^T I_i + \frac{\alpha}{2}\sigma_i^2 \; dt$$

where α is a scaling constant.

Adjoint Equations

For $i = 1, 2, \ldots, n$,

$$(8) \qquad \frac{d}{dt}\lambda_{1i} = -\frac{\partial H}{\partial S_i}$$

$$(9) \qquad \frac{d}{dt}\lambda_{2i} = -\frac{\partial H}{\partial I_i}$$

$$(10) \qquad \frac{d}{dt}\lambda_{3i} = -\frac{\partial H}{\partial R_i}$$

$$(11) \qquad \lambda_{1i}(T) = \lambda_{2i}(T) = \lambda_{3i}(T) = 0$$

$$(12) \qquad \frac{\partial H}{\partial \sigma_i} = 0$$ on interior of control sets.

Adjoint Equations (Cont.)

For $i = 1, 2, \ldots, n$, general forms of λ_{1i}', λ_{2i}' and λ_{1i}' are given as the following.

$$(13)\quad \frac{d}{dt}\lambda_{1i} = -\frac{\partial H}{\partial S_i}$$

$$= \lambda_{1i}(\beta_i I_i + \gamma\sigma_i + \sum_{k=1,k\neq i}^{n} a_{ik} + \mu_S) - \lambda_{2i}\beta_i I_i - \lambda_{3i}\gamma\sigma_i - \sum_{k=1,k\neq i}^{n} \lambda_{1k} a_{ik}$$

$$(14)\quad \frac{d}{dt}\lambda_{2i} = -\frac{\partial H}{\partial I_i}$$

$$= -1 - \lambda_{2i}(\beta_i S_i - \sum_{k=1,k\neq i}^{n} c_{ik} - \mu_I) + \lambda_{1i}\beta_i S_i - \sum_{k=1,k\neq i}^{n} \lambda_{2k} c_{ik}$$

$$(15)\quad \frac{d}{dt}\lambda_{3i} = -\frac{\partial H}{\partial R_i} = \lambda_{3i}(\sum_{k=1,k\neq i}^{n} a_{ik} + \mu_R) - \sum_{k=1,k\neq i}^{n} \lambda_{3k} a_{ik}$$

General form for the condition (12) is given by

$$(16) \quad \frac{\partial H}{\partial \sigma_i} = \alpha \sigma_i - \gamma S_i(\lambda_{1i} - \lambda_{3i}) = 0 \quad \text{at} \quad \sigma_i^*(t)$$

By solving (16) for $\sigma_i^*(t)$ for $i = 1, 2, \ldots, n$, we have

$$(17) \quad \sigma_i^*(t) = \frac{\gamma S_i(\lambda_{1i} - \lambda_{3i})}{\alpha}$$

subject to upper and lower bounds.

Numerical Results

Iterative Method

Guess for Optimal Control

Forward Sweep of SIR System

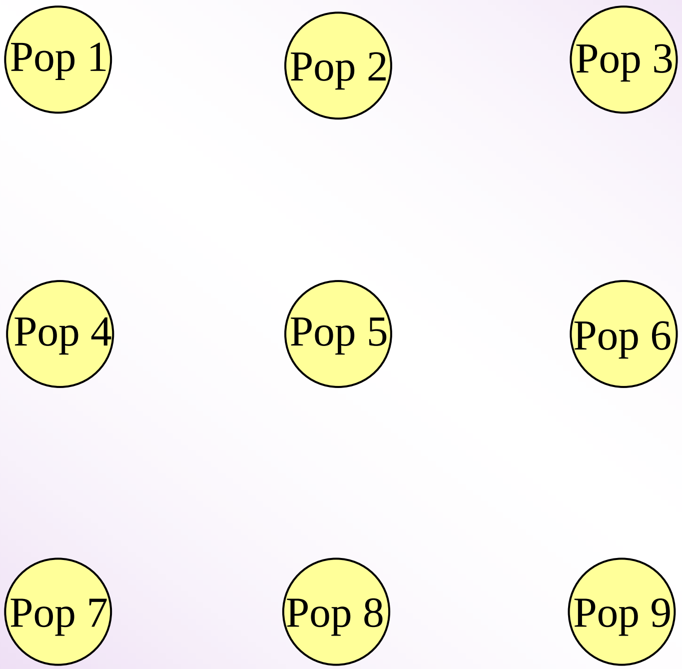
(4th order Runge-Kutta method)

Backward Sweep of Adjoint System

Update Control

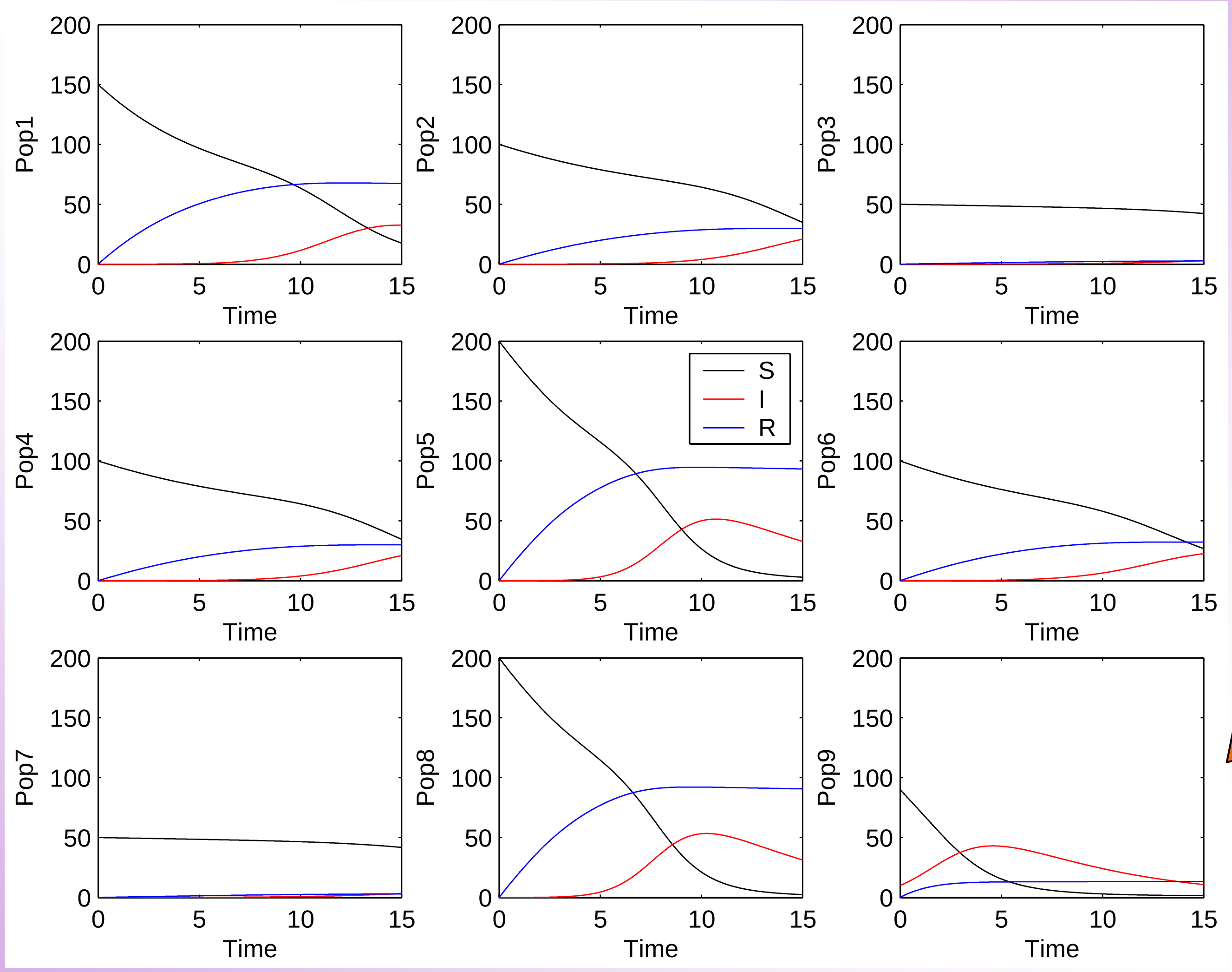
Repeat until successive iterates are close

Example: n = 9



Susceptibles, infecteds and removed

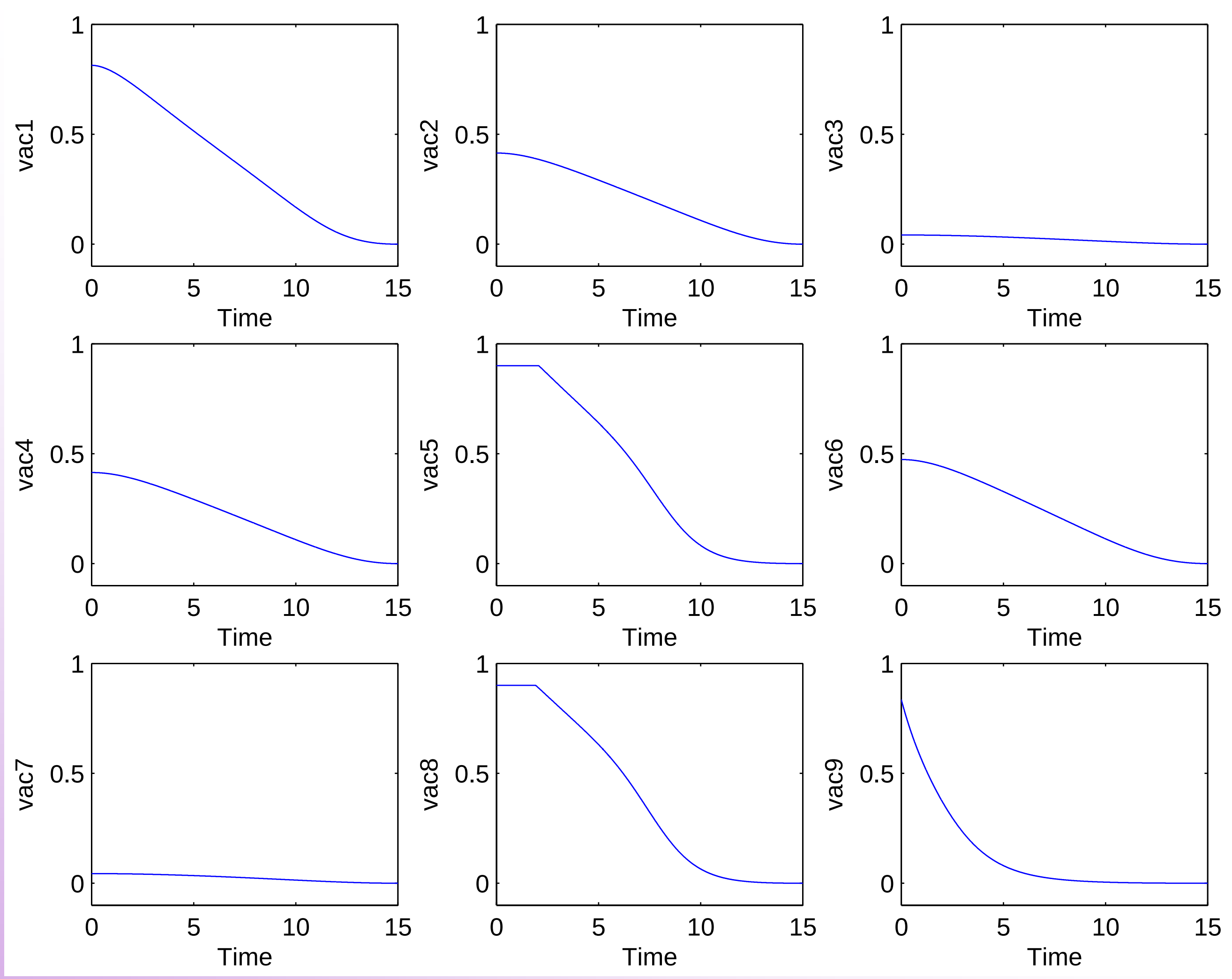
$\alpha = 100$, $\beta = 0.01$(same for all subpop.), $\gamma = 0.122$, $\mu_S = \mu_R = 0.00236$, $\mu_I = 0.1818$, $C = 1.5A$, $I_9(0) = 10(10\%)$, $S_1 = 150$, $S_2, S_4, S_6 = 100$, $S_3, S_7 = 50$.



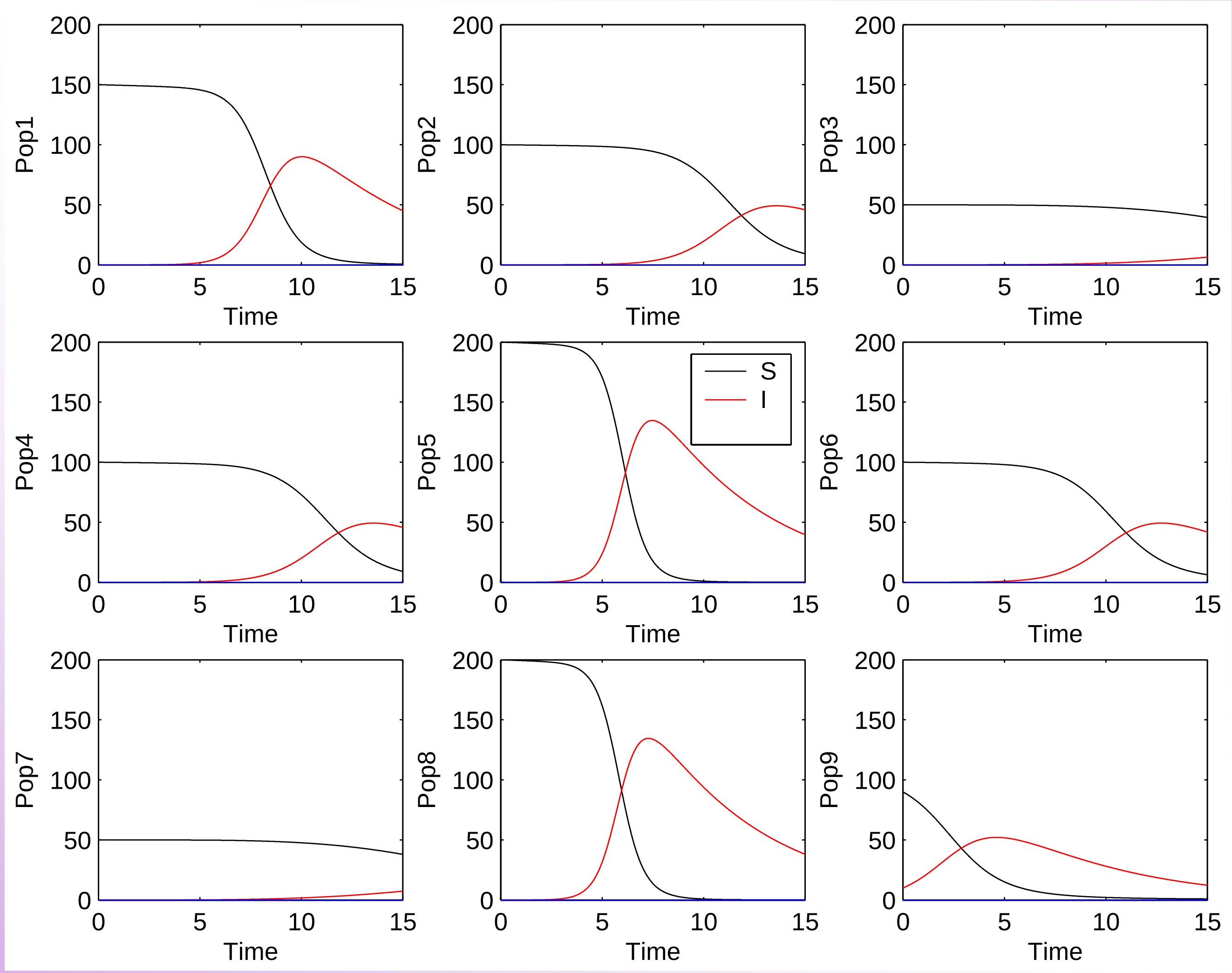
Initial Infecteds in pop 9.

Control(Vaccination)

α = 100, β = 0.01(same for all subpop.), γ = 0.122, $\mu_S = \mu_R$ = 0.00236, μ_I = 0.1818, C = 1.5A, $I_9(0)$ = 10(10%)

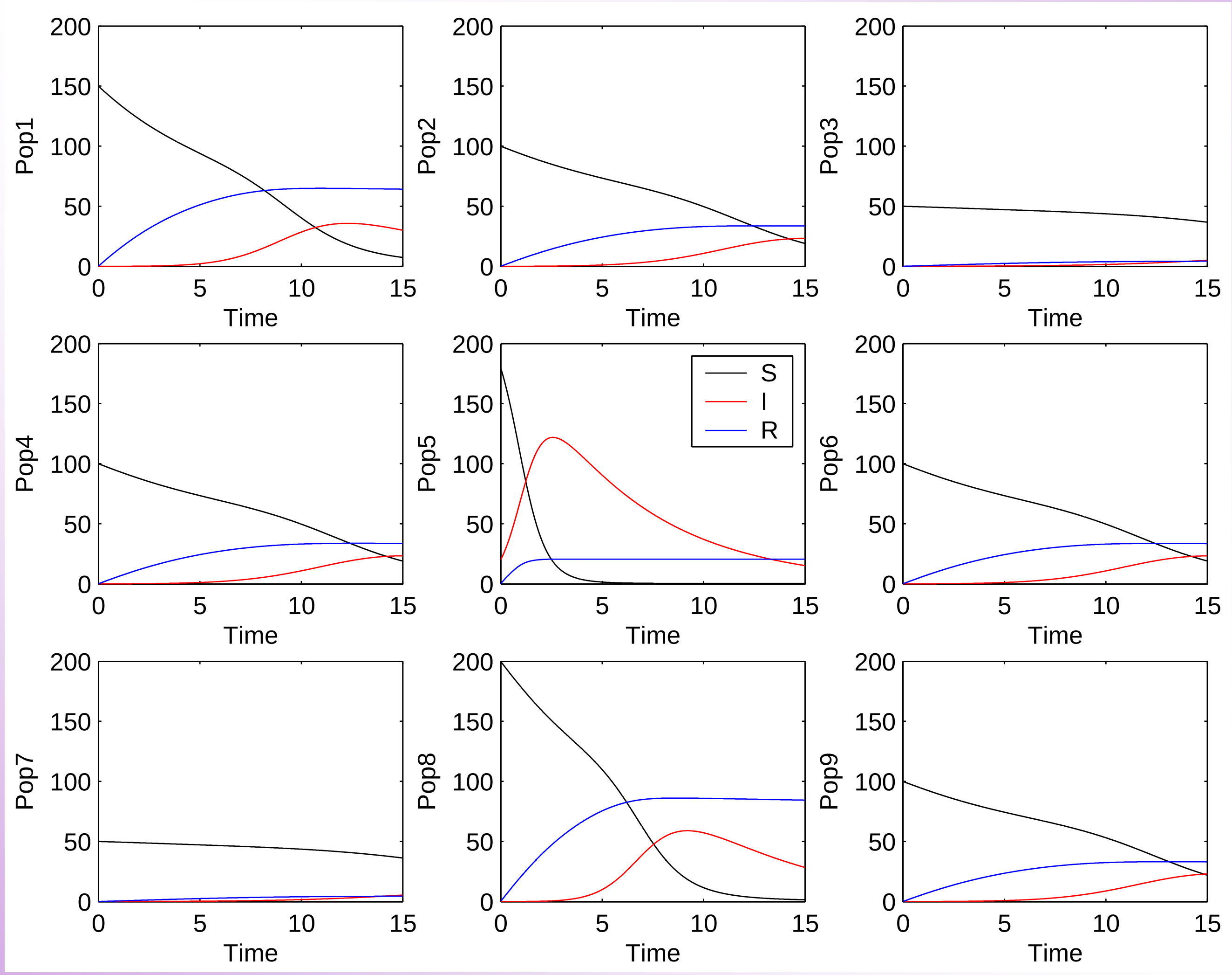


Reference –Susceptibles and infecteds with No Control (No Vaccination)



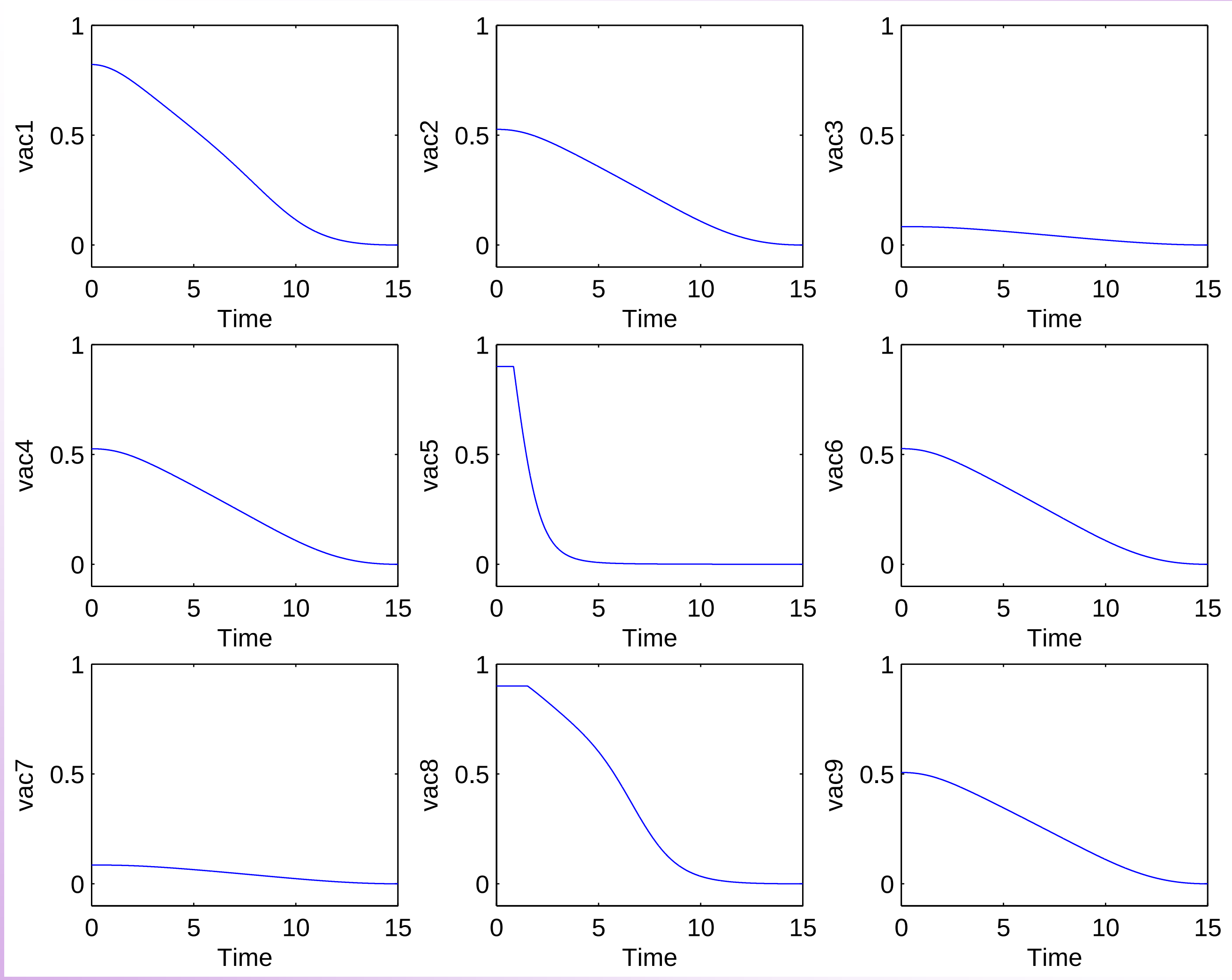
Susceptibles, infecteds and removed

$\alpha = 100$, $\beta = 0.01$(same for all subpop.), $\gamma = 0.122$, $\mu_S = \mu_R = 0.00236$, $\mu_I = 0.1818$, $C = 1.5A$, $I_5(0) = 20(10\%)$



Control(Vaccination)

α = 100, β = 0.01(same for all subpop.), γ = 0.122, $\mu_S = \mu_R$ = 0.00236, μ_I = 0.1818, $C = 1.5A$, $I_5(0)$ = 10(10%)



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